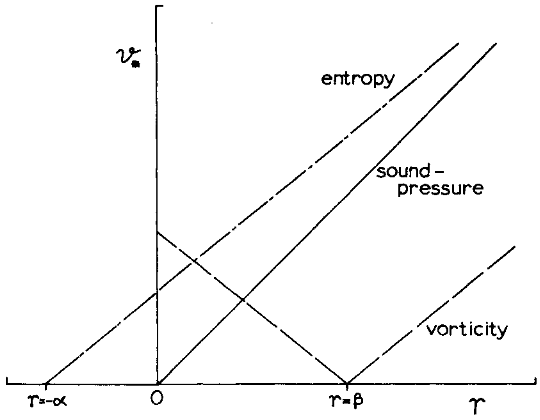

Fig. 4 Hot-wire mode diagrams for the three basic fluctuations.

varying degrees. Thus by means of hot-wire mode diagrams of the freestream and local model flows, it has been shown that in the gun tunnel a freestream sound-pressure fluctuation is "converted" into a multimode disturbance field on traversing an oblique shock wave.

This finding is in one sense consistent with a physical aspect of the current flow. The linearized theory of Kovasznay shows that the disturbance modes are noninteracting in a small perturbation field; that is, in the freestream and local model flows one mode may not give rise to another although they may coexist having derived from common or separate sources. Depending on its strength, within the plane of the shock wave the perturbations in the fluid properties may be large, which thereby allows for the possibility of the incident sound waves producing vorticity or entropy fluctuations at that location. Thus the shock wave becomes the "source" of the mixed disturbance field as detected by the hot-wire. Here mechanisms are not specified, although simple models may readily be postulated, in particular that for an incident pressure fluctuation giving rise to vorticity downstream of the shock wave.

High supersonic Mach number flows containing a superposition of significant disturbances including vorticity are difficult to analyze by means of hot-wire mode diagrams. Referring to Fig. 4, as $M \to \infty$, $\beta \to 2$, but even for modest hypersonic Mach numbers, say Mach 5, $\beta = 1.67$. Now the wire sensitivity ratio r is virtually synonymous with the wire overheating ratio, and for currently available high temperature wire alloys and for hypersonic flows with moderate stagnation temperatures ($T_o \sim 600°$K, for example) $r_{max} \sim 0.8$. Thus only a restricted range of r may be available over which must be determined the precise form of the mode diagram; moreover it is a curve which may have a minima at a value of r greater than the maximum achievable in practice. For higher temperature flows this restriction is even more severe. Gun tunnel flows, while being small perturbations, are inherently less steady than those generated in conventional wind tunnels, and scatter in the hot-wire data results. Therefore in the current tests no attempt was made to obtain sufficient data for the numerical resolution of the multimode flow over the inclined flat plate. Nevertheless, the least squares curves through four data points presented in Fig. 3 are of a form which clearly supports the main argument of this Note.

References

[1] Mabey, D. G., "The Influence of Flow Unsteadiness on Wind-Tunnel Measurements at Transonic Speeds," RAE Tech. Memo AERO 1473, Jan. 1973, Royal Aircraft Establishment, Bedford, England.

[2] Bergstrom, E. R. and Raghunathan, S., "Aerodynamic Noise and Boundary-Layer Transition Measurements in Supersonic Test Facilities," *AIAA Journal*, Vol. 10, No. 11, Nov. 1972, pp. 1531–1532.

[3] Moore, F. K., "Unsteady Oblique Interaction of a Shock Wave with a Plane Disturbance," Rept. 1165, 1954, NACA.

[4] Wagner, R. D., "Hot Wire Measurements of Freestream and Shock Layer Disturbances," *AIAA Journal*, Vol. 9, No. 12, Dec. 1971, pp. 2468–2470.

[5] Kovasznay, L. S. G., "Turbulence in Supersonic Flow," *Journal of the Aeronautical Sciences*, Vol. 20, No. 10, Oct. 1953, pp. 657–682.

[6] Laufer, J., "Aerodynamic Noise in Supersonic Wind Tunnels," *Journal of the Aerospace Sciences*, Vol. 28, No. 9, Sept. 1961, pp. 685–692.

Unsteady Flow Downstream of an Airfoil Oscillating in a Supersonic Stream

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Introduction

EXISTING closed form solutions for linearized, two-dimensional, supersonic flows past oscillating airfoils neglect the influence of the unsteady wake on the flowfield.[1] In such cases, it has still been possible to obtain the relevant information desired from solutions (i.e., airfoil pressure distributions and aerodynamic forces and moments) since the wakes only affect the flowfield downstream of the airfoils. Recently, the problem of the supersonic flow with subsonic axial velocity component past an oscillating cascade of airfoils has received serious attention.[2–5] For this flow geometry, the unsteady wakes influence the flowfield adjacent to blade surfaces and, hence, the wakes must be included in the analysis. This factor has posed a formidable obstacle in attempts to derive an exact solution and to date only approximate results have been obtained. With the purpose of providing further insight into this and other unsteady supersonic flow problems in which unsteady wakes must be considered, the flow past an oscillating airfoil (Fig. 1) has been re-examined. The solution in the region bounded by the airfoil leading and trailing edge Mach waves has been obtained previously by several investigators.[1,6] The solution for the unsteady flowfield downstream of the trailing edge Mach waves is described in this Note.

Analysis

All parameters discussed below are dimensionless. Lengths have been scaled with respect to blade chord, time with respect to blade chord divided by the freestream speed, and pressure with respect to the freestream density multiplied by one-half of the square of the freestream speed. The airfoil is assumed to be a flat plate performing rapid harmonic motions of small amplitude generally normal to the stream direction. These motions occur at a prescribed frequency ω. The unsteady wake is a thin vortex sheet which emanates from the trailing edge of the airfoil and extends infinitely far downstream. The small amplitude assumption permits the equations governing the unsteady flowfield to be linearized and leads to the additional simplification that boundary conditions, which apply on the airfoil and wake surfaces, can be satisfied on the mean position of these surfaces.

A modified velocity potential, $\psi(x, y)$, determined by the relation

$$\psi(x, y) = \phi(x, y, t) \exp[i(kMx - \omega t)] \tag{1}$$

is introduced to simplify the equations governing the unsteady flow. Here ϕ is the velocity potential as defined in the usual manner, $k = \omega M \mu^{-2}$, and M is the freestream Mach number. The modified potential must satisfy the differential equation[7]

$$\psi_{yy} - \mu^2 \psi_{xx} - \mu^2 k^2 \psi = 0 \tag{2}$$

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where $\mu^2 = M^2 - 1$. The pressure, $p(x, y, t)$, at a point in the flowfield is determined by

$$[p(x,y,t)-p_\infty]\exp[i(kMx-\omega t)] = P(x,y) = -2\left(\frac{\partial}{\partial x} - i\omega/\mu^2\right)\psi(x,y) \quad (3)$$

where p_∞ is the freestream pressure and $P(x, y)$ is the modified relative pressure.

The flow must satisfy the following boundary conditions. The fluid velocity component normal to the airfoil and wake surfaces, $v(x, y, t)$, must be continuous across these surfaces and the pressure must be continuous across the wake. The flow must be tangent to the airfoil surface, i.e.,

$$(\partial\psi/\partial y)(x,0^\pm) = V(x,0) = v(x,0,t)\exp[i(kMx-\omega t)] \quad 0 \leqq x \leqq 1 \quad (4)$$

where the normal velocity, $v(x, 0, t)$, and therefore $V(x, 0)$ is prescribed by the motion of the airfoil. Finally, there is no upstream propagation of disturbances and disturbances must be bounded at an infinite distance from their source.

Equation (2) and the condition of continuous normal velocity across the airfoil and its wake indicate that ψ must be an odd function of y. Furthermore, since $\psi = 0$ for $x < 0$, it is only necessary to determine a solution in the first quadrant of the xy-plane. If $V(x, 0)$ is assumed to be known for $x > 0$, then a Laplace transform approach yields the following result for the modified potential[1,7]

$$\psi(x,y) = -\mu^{-1}\int_0^{x-\mu y} V(\xi,0)J_0(kR)\,d\xi \quad x > \mu y > 0 \quad (5)$$

where J_0 is the Bessel function of the first kind of order zero and $R = [(x-\xi)^2 - \mu^2 y^2]^{1/2}$. Expressions for the modified normal velocity or upwash and the fluid pressure follow from Eqs. (3–5) and have the form

$$V(x,y) = V(x-\mu y,0) - k\mu y\int_0^{x-\mu y} V(\xi,0)R^{-1}J_1(kR)\,d\xi \quad x > \mu y > 0 \quad (6)$$

$$P(x,y) = 2\mu^{-1}V(x-\mu y,0) - 2\mu^{-1}\int_0^{x-\mu y} V(\xi,0)\times [k(x-\xi)J_1(kR)/R + i\omega\mu^{-2}J_0(kR)]\,d\xi \quad x > \mu y > 0 \quad (7)$$

Since disturbances propagate downstream, the fact that the normal velocity has not been specified on the wake surface does not affect the solution in the region $x \leqq 1 + \mu|y|$ (see Fig. 1). Therefore, the foregoing results provide the necessary information for determining the pressure distributions and thus the aerodynamic force and moment acting on the airfoil. The analyses of Refs. 1 and 6 were essentially completed at this point.

Since ψ is an odd function of y, it follows from Eq. (3) that $(p - p_\infty)$ is an odd function of y and, therefore, the magnitude of the pressure difference across the wake is $2|p(x, 0^+, t)|$. To satisfy the boundary condition of pressure continuity across the wake, it is clear that the pressure at a point on the wake surface must be equal to its freestream value, i.e.,

$$[p(x,0,t)-p_\infty]\exp[-i(kMx-\omega t)] = P(x,0) = 0, \quad x > 1 \quad (8)$$

An expression for the wake upwash may be derived from Eqs. (7) and (8) and has the form

$$V(x,0) = \int_1^x V(\xi,0)\{kJ_1[k(x-\xi)] + i\omega\mu^{-2}J_0[k(x-\xi)]\}\,d\xi + F(x) \quad x > 1 \quad (9)$$

where

$$F(x) = \int_0^1 V(\xi,0^+)\{kJ_1[k(x-\xi)] + i\omega\mu^{-2}J_0[k(x-\xi)]\}\,d\xi \quad (10)$$

The foregoing Volterra integral equation may be solved numerically to determine the wake upwash distribution. Once the latter is known, the modified potential and pressure at any point in the flowfield can be determined from Eqs. (5) and (7).

A more convenient approach for determining the flowfield downstream of the trailing edge Mach waves follows after recasting the boundary value problem for the modified potential in terms of its known values on the blade and wake surfaces. The modified potential on the blade surface is obtained from Eq. (5) and the prescribed airfoil motion, i.e.,

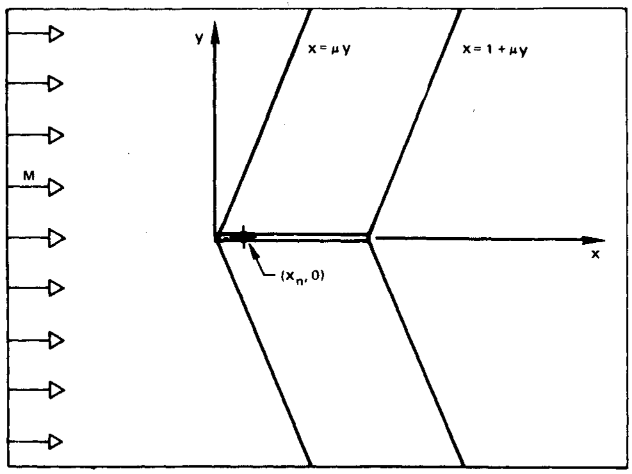

Fig. 1 Supersonic flow past an oscillating airfoil.

$$\psi(x,0^+) = -\mu^{-1}\int_0^x V(\xi,0)J_0[k(x-\xi)]\,d\xi \quad 0 \leqq x \leqq 1 \quad (11)$$

Values on the wake surface are determined by integration of Eq. (3) subject to the condition $P(x, 0) = 0$. It follows that

$$\psi(x,0^+) = \psi(1,0^+)\exp[i\omega\mu^{-2}(x-1)] \quad x \geqq 1 \quad (12)$$

The bound circulation of the airfoil is given by $2\psi(1,0^+) \times \exp[i(\omega t - kM)]$. Physically, condition (12) implies that counter vortex elements are shed from the trailing edge of the airfoil at a rate equal to the time variation of the bound circulation and convected downstream at the freestream speed.[7] The potential for the entire flowfield may now be expressed in terms of the known boundary values of ψ by again utilizing the Laplace transform procedure.

The transform

$$\bar{\psi}(s,y) = \int_0^\infty \psi(x,y)\exp(-sx)\,dx \quad (13)$$

of Eq. (2) is

$$(d^2\bar{\psi}/dy^2) - \mu^2\lambda^2\bar{\psi} = 0, \quad \lambda^2 = (s^2+k^2) \quad (14)$$

Equation (9) has the general solution

$$\bar{\psi}(s,y) = A(s)\exp(-\mu\lambda y) + B(s)\exp(\mu\lambda y) \quad y > 0 \quad (15)$$

In order to satisfy the requirement of no upstream propagation of disturbances, the branch of λ for which the real part of λ is greater than zero, must be chosen. Then since disturbances must be bounded as $y \to \infty$.

$$B(s) = 0 \quad (16)$$

and therefore

$$A(s) = \lim_{y\to 0^+}\bar{\psi}(s,y) = \bar{\psi}(s,0^+) \quad (17)$$

It follows after inverting Eq. (15) that[8]

$$\psi(x,y) = \psi(x-\mu y,0^+) - k\mu y\int_0^{x-\mu y}\psi(\xi,0^+)J_1(kR)R^{-1}\,d\xi \quad (18)$$

An expression for the modified normal velocity distribution on the wake follows after differentiating the foregoing result with respect to y and letting y approach zero.

$$v(x,0,t)\exp[i(kMx-\omega t)] = V(x,0) = -\omega\mu^{-1}\psi(1,0^+)\exp[i\omega\mu^{-2}(x-1)] - \mu k\int_0^x \psi(\xi,0^+)(x-\xi)^{-1}J_1[k(x-\xi)]\,d\xi \quad x > 1 \quad (19)$$

where $\psi(\xi, 0^+)$ is determined from Eq. (11) for $0 \leqq \xi \leqq 1$.

In general, the amplitude and spatial frequency of the wake upwash distribution increases with increasing frequency of the

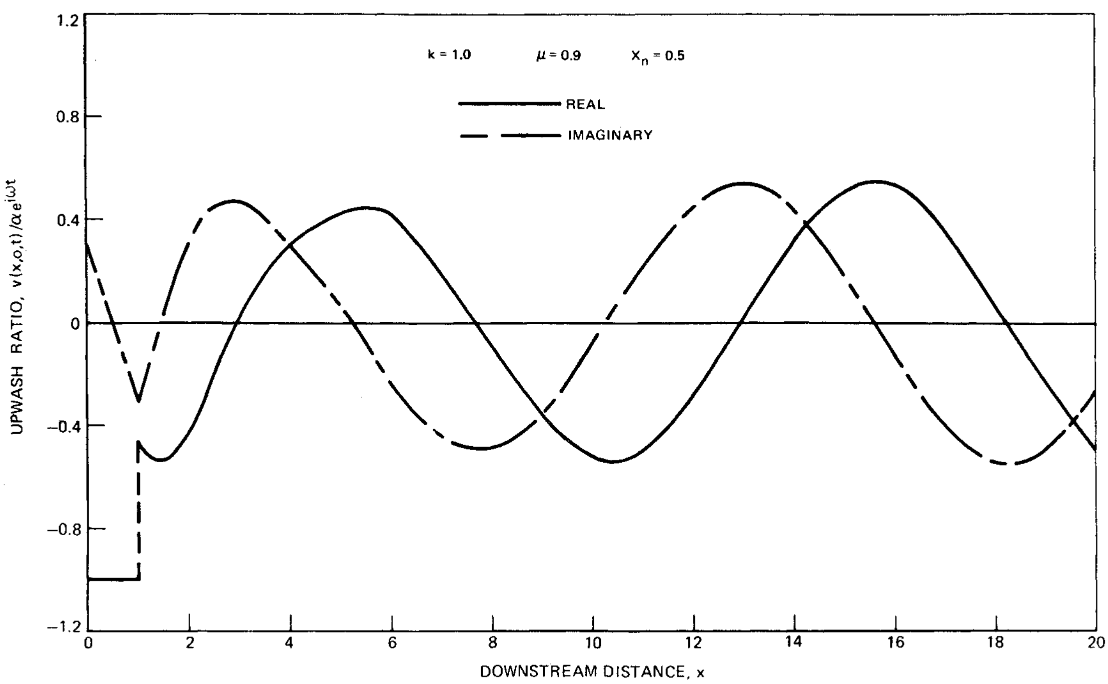

Fig. 2 Upwash on airfoil and wake for a pitching motion.

blade motion. At very low frequencies, the first term on the right-hand side of Eq. (19) dominates, at least in the near wake. This term arises from the counter vorticity shed from the trailing edge of the airfoil and it provides an upwash distribution, $v(x, 0, t)$, having a spatial frequency equal to ω. Additional upwash frequency components arise from the second term in Eq. (19). In particular, the upwash has spatial frequencies approximately equal to $\omega M(M+1)^{-1}$ and $\omega M(M-1)^{-1}$, which correspond to disturbances moving downstream at speeds equal to the freestream speed plus and minus the speed of sound propagation, respectively. The ratio of the wake upwash to airfoil angular displacement is plotted vs distance in Fig. 2 for an airfoil undergoing a pitching motion, $\alpha e^{i\omega t}$, about an axis located at the point $(x_n, 0)$. The predominant spatial frequency of the waves depicted in this figure is the vortex shedding frequency ω.

Once the normal velocity distribution on the airfoil wake is determined, the pressure field downstream of the airfoil can be obtained from Eqs. (3) and (7). Calculations have revealed that the wake upwash exerts an important influence on this pressure field even at points which are of the order of ten chord lengths from the wake.

Conclusions

The closed form solution for the unsteady flowfield produced by an airfoil undergoing harmonic motions in a supersonic stream has been extended to include the region downstream of the airfoil. The essential information required to complete the determination of the unsteady field was the wake pressure distribution. For an oscillating airfoil in a subsonic stream, the wake must be considered to specify uniquely the pressure distribution on the airfoil. This is not the case in supersonic flow. However, if the oscillating airfoil problem is solved in terms of prescribed normal velocities on the airfoil surface and if the upwash in the wake is neglected, a pressure discontinuity will appear at the trailing edge of the airfoil and extend downstream. In addition, large errors will be present in the computed pressure field downstream of the airfoil. These factors are important considerations in aerodynamic interference problems and signal the need for a vortex sheet representing the unsteady wake in potential supersonic flow.

References

[1] Miles, J. W., *The Potential Theory of Unsteady Supersonic Flow*, Cambridge University Press, Cambridge, England, 1959, pp. 49–60.

[2] Snyder, L. E., "Supersonic Unstalled Flutter," *Aeroelasticity in Turbomachinery, Proceedings of a Project Squid Workshop held at Detroit Diesel Allison*, edited by S. Fleeter, Office of Naval Research, June 1972, pp. 164–195.

[3] Kurosaka, M., "On the Unsteady Supersonic Cascade with a Subsonic Leading Edge—An Exact First Order Theory—Part 1," Paper 73-GT-15, ASME Gas Turbine Conference and Products Show, Washington, D.C., 1973.

[4] Kurosaka, M., "On the Unsteady Supersonic Cascade with a Subsonic Leading Edge—An Exact First Order Theory—Part 2," Paper 73-GT-16, ASME Gas Turbine Conference and Products Show, Washington, D.C., 1973.

[5] Verdon, J. M., "The Unsteady Aerodynamics of a Finite Supersonic Cascade with Subsonic Axial Flow," *Transactions of the ASME, Journal of Applied Mechanics*, Ser. E, Vol. 40, No. 3, Sept. 1973, pp. 667–671.

[6] Garrick, I. E. and Rubinow, S. I., "Flutter and Oscillating Air-Force Calculations for an Airfoil in a Two-Dimensional Supersonic Flow," TN 1158, 1946, NACA.

[7] Ashley, H. and Landahl, M., *Aerodynamics of Wings and Bodies*, Addison-Wesley, Reading, Mass., 1965, pp. 248, 250, 257.

[8] Scott, E. J., *Transform Calculus*, Harper and Brothers, New York, 1955, pp. 72, 75, 323.

Higher Order Boundary Layer for Viscous Flow past Sharp Wedges

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Introduction

TWO decades have been devoted to finding a second term in the asymptotic expansion of the Navier-Stokes equation for the flat plate case.[1,2,3] They concluded that the second nonzero

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